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Disordered Spherical Model

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The most general expression of the free energy in the disordered spherical model is obtained. Based on this expression the following are shown. (a) The ferromagnetic order in the translationally invariant spherical model is unstable against an arbitrarily small random field if $d \leqslant 4$. (b) Straightforward generalization of the spherical model to the disordered case for a finite-range interaction has some rather unnatural properties: the phase transition in the model exists even in one dimension, and even in the case of ferromagnetic interaction it does not vanish as a homogeneous external field is switched on and spontaneous magnetization is zero for $T < T_c$. (c) For the ferromagnetic interaction, a modification of the disordered spherical model is proposed which does not have such properties and displays the behavior expected for the disordered ferromagnets. The paper also discusses the role of fluctuation (cluster) effects and the structure of the spontaneous magnetization field for the disordered spherical model. The results essentially rest upon the spectral properties of random self-adjoint operators obtained by the author earlier and in the present paper.

KEY WORDS: Spherical model; disordered systems; phase transitions; random operators.

1. INTRODUCTION

The past decade was characterized by rising interest in various disordered systems, disordered spin systems in particular. The results obtained for such systems have become at present a significant part of the magnetic phenomena theory. This in particular has been so because the study of such systems turned out to be closely related to certain fundamental problems of physics of magnetism. However, despite the many interesting suggestions as well as approximate and numerical calculations of this period, the theory of quite a

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lot of phenomena remains essentially unclear. This primarily refers to phase transitions and critical phenomena which may take place in disordered spin systems, especially those where the interaction between spins is not positive. Among the latter, the most interesting are those in which the proportions of positive and negative values of the exchange integral are approximately equal, since, as is widely accepted, there may appear therein a low-temperature phase peculiar to disordered systems, namely, the spin glass.

Yet, nontrivial, exactly solvable models and rigorous results in this area are rather few, numbering hardly more than some molecular field-type models. It is therefore reasonable to use generalization to the disordered case of the spherical model[1], one of the simplest models for magnets, which, unlike the Ising model whence it was in a sense deduced, admits an exact solution in the ordered (translationally invariant) case for any interactions and in all dimensions d of space, and when $d \geqslant 3$ displays the phase transition (see, e.g., a review, Ref. 2), although with critical exponents somewhat differing from those in more realistic models.[(3)] One of such generalizations, which was designed to describe the phase transition to the spin glass state, was proposed in Ref. 4. This model has a phase transition with many rather attractive properties. However, this model is in fact a hybrid of the spherical and molecular field-type models, for besides the spherical condition $\sum_{r \in V} s_r^2 = N$ replacing the Ising condition $s_r^2 = 1$, this model assumes in fact that the range of spin interaction equals infinity.

In the present work we shall arrive at the expression for the free energy of the spherical model without the latter assumption, that is, we shall consider the disordered spherical model with a finite interaction range, as is common in statistical physics.

However, an analysis of the resulting expression shows the model to have rather unusual thermodynamical properties. Namely, the phase transition in this model is possible even in the one-dimensional case, but nevertheless, even if the interaction is ferromagnetic, there arises no spontaneous magnetization, and the phase transition itself does not vanish as a homogeneous external field is switched on. These properties, which seem rather unnatural in the present-day views and as against the ordered spherical model,[2] are due to the fact that, as we show, thermodynamics of the spherical model of the most general form is essentially determined by the structure of the spectrum of the exchange integral matrix $\mathcal{J}_{rr'}$ in the neighborhood of its upper boundary $\mathcal{J}$, by the behavior of the density of states $\rho(a)$ of this matrix near $\mathcal{J}$ in particular. But as is well known in the electronic disordered systems theory,[(18–21)] this behavior is quite different

[2] Similar results were also obtained in the series of papers, Ref. 8, where quite a different approach was used to study the so-called mean spherical model for the special case of quasi-one-dimensional (layered) disorder (for details see Appendix A).

in the ordered and disordered cases. Namely, in the former, $\rho(a) \sim (\mathcal{J} - a)^{d/2-1}$ when $a \uparrow \mathcal{J}$, and in the latter, $\rho(a)$ tends to zero exponentially when $a \uparrow \mathcal{J}$ [e.g., as $\ln \rho(a) \sim -(\mathcal{J} - a)^{-d/2}$ (Refs. 18–20)]. This, in turn, is caused by fluctuation (cluster) effects in disordered systems and the closely associated existence of the point spectrum in the vicinity of $\mathcal{J}$ (the so-called localization of eigenfunctions in infinite disordered systems[(21)]). From the same point of view it is clear why the results of this work for the disordered spherical model with finite interaction range are different from what they are in the infinite interaction range model.[(4)] The reason is that the transition to the infinite range case induces the largest changes in the spectrum structure near $\mathcal{J}$. This fact is also well known in the electron-disordered system theory where the self-consistent field-type approximations (coherent potential approximation, modified propagator approximation, etc.[(6,7)]) are normally used only for the calculation of spectrum-integrated quantities and result in behaviors of the density of states in the vicinity of spectrum boundaries that are untrue to reality.

All the above arguments suggest that direct generalization of the spherical model to the disordered case, that is, mere replacement of the $r - r'$-dependent exchange integrals $\mathcal{J}_{r-r'}$ by random variables $\mathcal{J}_{rr'}$, is largely an inadequate approach to disordered magnets. The present paper proposes a modification of this model which in the ordered case is thermodynamically equivalent to the known one,[(1)] and in the disordered case and for positive exchange integrals[3] turns out a good model of ferromagnets, since the phase transition exists there only at $d \geqslant 3$ and vanishes if a uniform field is switched on, and the critical exponents are the same as those in the ordered spherical model.

The contents of this paper are briefly as follows.

In Section 2 an expression is derived for the free energy of the spherical model for a most general situation where random exchange integrals and the external field satisfy only the conditions of spatial homogeneity in the mean, disappearance of statistical correlations at infinitely distant points, and finiteness of the interaction range. In doing so, because the traditional procedure based on the steepest-descent method[(1,2)] encounters some difficulties in the disordered case, we use a different calculation procedure for the free energy in the spherical model which is closely related to the general statistical physical idea of equivalency of various ensembles in the thermodynamical limit.

The first part of Section 3 is a brief discussion of the partly known properties of the ordered spherical model, with a special emphasis on the usually ignored case of oscillating exchange integrals, since in connection

[3] The modification of the spherical model for the oscillating interaction case will be considered elsewhere.

with the spin glass problem, competing interactions are now attracting great interest in physics of disordered spin systems. Therefore it is desirable to understand to what extent the spin glass features become apparent in the ordered case.

In the second part of Section 3 the spherical model with a random external field is examined. The critical properties of ordered magnetic systems placed in random external fields are now under active study (see, e.g., Refs. 9–11), both because they are the simplest variant of disordered spin systems, and because to such models sometimes models with random exchange in a homogeneous field may be reduced.[11,13–15] We show in particular that the ferromagnetic order is unstable if a small random field of rather arbitrary nature is switched on, provided that $d \leqslant 4$ (cf. Refs. 9 and 16).

In Section 4, the spherical model with random exchange integrals is considered. In the first part of the section the above properties of disordered spherical model are proved (the existence of the phase transition in the one-dimensional case, etc.). In doing so, we use some properties of the random operator spectrum structure, proved in Appendixes A–C. In particular, in Appendix A the fluctuation asymptotic formulas for the density of states are obtained by a method which explicitly uses the existence of cluster phenomena in the problem under consideration.

The second part of the section contains a description of the above-mentioned modification of the disordered spherical model and demonstrates that it leads to plausible physical results.

In the last section we discuss the structure of spontaneous magnetization m_r, which is equal to the Gibbs mean of the spin variable s_r and in a disordered system is a random field. The reasons are analyzed of the absence of spontaneous magnetization in the disordered spherical model, and are found to be associated with the structure of the eigenfunctions of random operators at the spectrum boundary that is quite different from the structure in the ordered case. The reasoning of this section is of more heuristic character and in part relies upon ideas of the spectrum structure of some random operator classes which are widely accepted though not yet proved for the non-one-dimensional case.

It might be also proper to emphasize that the results of this work are of thermodynamical nature. The structure of the corresponding Gibbs field and the states in the disordered spherical model will be considered elsewhere.

2. FREE ENERGY OF THE DISORDERED SPHERICAL MODEL

Consider a d-dimensional lattice $\mathbb{Z}^d$ of vectors r with integer components $n_1, \ldots, n_d$ and a parallelepiped V in it consisting of vectors with

$|n_i| \leqslant n$, $i = 1, \ldots, d$. As is known, the spherical model states are sets $\{s_r : r \in V, \sum_{r \in V} s_r^2 = N\}$, $N = (2n+1)^d$. The energy of each set is

$$H_N = -\frac{1}{2} \sum_{r,r' \in V} \mathcal{J}_{rr'} s_r s_{r'} - \sum_{r \in V} s_r h_r \tag{2.1}$$

or more compactly

$$H_N = -\tfrac{1}{2}(\hat{\mathcal{J}}_N s, s) - (h, s) \tag{2.2}$$

In the disordered case, the exchange integrals $\mathcal{J}_{rr'}$ and the external field h_r are random variables. In this section we assume that they satisfy the following general conditions:

(1) $|\mathcal{J}_{rr'}|$ and $|h_r|$ are bounded uniformly in $r, r' \in \mathbb{Z}^d$ and all realizations.[4]

(2) For any $a \in \mathbb{Z}^d$, expectations of form[5]

$$\langle \mathcal{J}_{r_1+a, r_2+a} \cdots \mathcal{J}_{r_{2M-1}+a, r_{2M}+a} h_{r_{2M+1}+a} \cdots h_{r_{2M+1}+a} \rangle$$

are independent of a for any $r_1, \ldots, r_{2M+P}$, M and P.

(3) For $a \to \infty$, expectations of form

$$\langle \mathcal{J}_{r_1, r_2} \cdots \mathcal{J}_{r_{2M-1}, r_{2M}} h_{r_{2M+1}} \cdots h_{r_{2M+P}} \mathcal{J}_{r'_1+a, r'_2+a} \cdots \mathcal{J}_{r'_{2Q-1}+a, r'_{2Q}+a} h_{r'_{2Q+1}+a} \cdots h_{r'_{2Q+R}+a} \rangle$$

are factorized into a product of mean values

$$\langle \mathcal{J}_{r_1 r_2} \cdots h_{r_{2M+P}} \rangle \langle \mathcal{J}_{r'_1 r'_2} \cdots h_{r'_{2Q+R}} \rangle$$

(4) $\mathcal{J}_{r,r'} = 0$ if $|r - r'| > R_0$, $R_0 < \infty$.

Conditions (2)–(4) express the properties of macroscopic spatial homogeneity of the system, disappearance of statistical correlations in infinitely distant points, and finiteness of the interaction range.

A more general and formalized variant of these conditions may be also given. Assume that in set Ω of elementary events ω, in which random quantities $\mathcal{J}_{rr'}$ and h_r are defined (and thus are functions of ω), a group of transformations T_a, $a \in \mathbb{Z}^d$ is defined such that

$$\mathcal{J}_{rr'}(T_a\omega) = \mathcal{J}_{r+a, r'+a}(\omega), \qquad h_r(T_a\omega) = h_{r+a}(\omega) \tag{2.3}$$

Then condition (2) means that transformations T_a preserve the probabilities of any events (measurable sets) in Ω, and condition (3) means that group T_a

[4] We have recourse to these assumptions in order to simplify the subsequent discussion. The results of this section are valid even under the assumption of finiteness of several first moments of quantities $|\mathcal{J}_{rr'}|$ and $|h_r|$.

[5] Henceforth the symbol $\langle \cdots \rangle$ denotes averaging over the realizations of the involved random variables.

has the property of mixing[6] and therefore in particular has no nontrivial invariant subspaces.(22)

Our object in this section is to calculate the free energy in the macroscopic limit $N \to \infty$, i.e.,

$$f = -\lim_{N\to\infty} (\beta N)^{-1} \ln Z_N$$
$$Z_N = \int_{\sum_{r\in V} s_2^2 = N} e^{-\beta H_N} \prod_{r\in V} ds_r \tag{2.4}$$

The traditional way to calculate the free energy in the spherical model is based on the steepest descent method.(1,2) However, in the disordered case, where exchange integrals are not functions of difference $r - r'$, the application of the steepest-descent method is difficult because of lack of information on the distribution of the eigenvalues of matrix $\mathcal{J}_{rr'}$ with elements $\mathcal{J}_{r,r'}$ $r \neq r' \in V$ adjacent to the spectrum right-hand boundary for $N \to \infty$. Therefore we use another approach, based entirely on the real analysis, which is similar to that used in Ref. 24 to prove the existence of the thermodynamical limit in the microcanonical ensemble from that of the thermodynamical limit in the canonical ensemble (see also Ref. 56).

Let

$$\mathcal{J} = \sup_{\|x\|=1} \sum_{r\neq r'\in\mathbb{Z}^d} \mathcal{J}_{rr'} x_r \bar{x}_{r'}, \qquad \|x\|^2 = \sum_{r\in\mathbb{Z}^d} |x_r|^2$$

Clearly, $\mathcal{J} \geqslant \mathcal{J}_N$, where $\mathcal{J}_N$ is the maximum eigenvalue of matrix $\hat{\mathcal{J}}_N$ with elements $\mathcal{J}_{rr'}(1-\delta_{rr'})$, $r, r' \in V$ and therefore matrix $\hat{A}_N = \mathcal{J} - \hat{\mathcal{J}}_N$ is nonnegatively defined for all N. Obviously,

$$f = \psi - \mathcal{J}/2 \tag{2.5}$$

where ψ is the free energy for the partition function

$$Q_N = \int_{\sum_{r\in V} s_r^2 = N} \exp\left[-\frac{\beta}{2}(\hat{A}_N s, s) + \beta(h,s)\right] \prod_{r\in V} ds_r$$

Along with Q_N, consider an integral of the similar expression for an N-dimensional ball $B_{N,u}$ of radius $N^{1/2}u$:

$$Q_{N,u} = \int_{B_{N,u}} \exp\left[-\frac{\beta}{2}(\hat{A}_N s, s) + \beta(h,s)\right] \prod_{r\in V} ds_r \tag{2.6}$$

and let

$$\chi(u) = \lim_{N\to\infty} \chi_N(u) = -\lim_{N\to\infty} (\beta N)^{-1} \ln Q_{N,u} \tag{2.7}$$

[6] It is enough to assume that group T_a is ergodic, which corresponds to the limit in conditions (3) in Cesaro's sense.(22)

From obvious inequality $N^{-1/2}Q_{N,1} \leqslant Q_N \leqslant N^{-1/2}(u-1)Q_{N,u}$ it follows that

$$\lim_{u \downarrow 1} \chi(u) \leqslant \psi \leqslant \chi(1) \tag{2.8}$$

By substituting $us_r \to s_r$ in Eq. (2.6) and calculating then the logarithmic derivative, find, with allowance for the condition (1), that derivative $\partial\chi_N/\partial u$ is uniformly bounded in N and $0 < u_0 \leqslant u \leqslant u_1 < \infty$. Therefore if the limit function $\chi(u)$ exists, it is continuous, and then it follows from Eq. (2.8) that

$$\psi = \chi(1) \tag{2.9}$$

Thus the problem is reduced to determining $\chi(1)$ specified by Eqs. (2.6) and (2.7).

To find it, consider the following Gaussian integral:

$$\Xi_N = \int_{\mathbb{R}^N} \exp\left\{ -\frac{\beta}{2}\left[(2\zeta + \hat{A}_N)s, s\right] + \beta(h,s)\right\} \prod_{r \in V} ds_r \tag{2.10}$$

where $\zeta > 0$. Evidently,

$$\Xi_N = \left(\frac{2\pi}{\beta}\right)^{N/2} \det\left[2\zeta + \hat{A}_N\right]^{-1/2} \exp\frac{\beta}{2}(\hat{G}_N h, h) \tag{2.11}$$

where $\hat{G}_N = (2\zeta + \hat{A}_N)^{-1}$. On the other hand, by integrating by parts in Eq. (2.11), we have

$$\Xi_N = \beta\zeta u \int_0^\infty du \exp\left\{-\beta N\left[\chi_N(u) + \zeta u\right]\right\} \tag{2.12}$$

But when $N \to \infty$ with probability 1, there exists

$$-\lim_{N\to\infty} (\beta N)^{-1} \ln \Xi_N = f_G(\zeta) \tag{2.13}$$

Indeed, it follows from Eq. (2.11) that the left-hand side of this equation is

$$-\frac{1}{2\beta}\ln\frac{2\pi}{\beta} + \frac{1}{2\beta}\int_0^\infty \ln(2\zeta + a)\, d\nu_N(a) - \frac{1}{2}\int_0^\infty \frac{d\nu_N^{(1)}(a)}{2\zeta + a} \tag{2.14}$$

Here $\nu_N(a)$ and $\nu_N^{(1)}(a)$ are nondecreasing functions specified by relations

$$\nu_N(a) = N^{-1}\sum_{a_k \leqslant a} 1, \qquad \nu_N^{(1)}(a) = N^{-1}\sum_{a_k \leqslant a}\left|\sum_{r\in V}\psi_k(r)h_r\right|^2 \tag{2.15}$$

and a_k, $\psi_k(r)$, $k = 1, \ldots, N$ are the eigenvalues and eigenfunctions of matrix $\hat{A}_N$.

As was shown in Ref. 25, under conditions (1)–(4), there is such a nonrandom nondecreasing function $\nu(a)$, that at all its continuity points, with probability 1,

$$\lim_{N\to\infty} \nu_N(a) = \nu(a) \tag{2.16}$$

(see footnote 7). By generalizing the arguments in Ref. 25, one can also show that under the same conditions, a similar statement is true for sequence $\nu_N^{(1)}(a)$ as well:

$$\lim_{N\to\infty} \nu_N^{(1)}(a) = \nu_1(a) \tag{2.17}$$

(see footnote 8). Since in Eq. (2.14), $\zeta > 0$, $\nu_N(\infty) = \nu_N^{(1)}(\infty) = 1$, then in this expression we can go to the limit under both the integrals and obtain Eq. (2.13) with

$$f_G(\zeta) = -(2\beta)^{-1} \ln \frac{2\pi}{\beta} + f_1(\zeta) + f_2(\zeta) \tag{2.18}$$

where

$$f_1(\zeta) = \frac{1}{2\beta} \int_0^\infty \ln(2\zeta + a)\, d\nu(a),$$

$$f_2(\zeta) = -\frac{1}{2} \int_0^\infty \frac{d\nu_1(a)}{2\zeta + a} = -\lim_{N\to\infty} (2N)^{-1} (\hat{G}_N h, h)$$

Now in order to prove the existence of the thermodynamical limit in the spherical model we deal with, we shall use the following fact.

Let nondecreasing concave function $\Phi_N(\zeta)$ and nonincreasing function $\varphi_N(u)$, $u \geqslant 0$ be related as

$$\exp[-N\Phi_N(\zeta)] = \int_0^\infty du \exp\{-N[\varphi_N(u) + \zeta u]\}$$

Besides, let $\Phi_N(\zeta)$ tend to the limit continuously differentiable function $\Phi(\zeta)$ for every $\zeta > 0$, when $N \to \infty$. Then $\varphi_N(u)$ also tends to the limit $\varphi(u)$ for every $u > 0$, and

$$\varphi(u) = \begin{cases} \sup\limits_{\zeta > 0} \{\Phi(\zeta) - \zeta u\} & \text{if } u \leqslant \Phi'(+0) \\ \Phi(+0) & \text{if } u \geqslant \Phi'(+0) \end{cases}$$

This statement is an improvement of the Tauberian theorem for the Laplace transform of Ref. 24. Its proof does not differ significantly from that of the above-mentioned theorem, and we shall omit it.

[7] The derivative of function $\nu(a)$, when existent, is termed the density of states of the random operator $\hat{\mathcal{J}}$ determined by matrix $\mathcal{J}\delta_{rr'} + \mathcal{J}_{rr'}(1 - \delta_{rr'})$, $r, r' \in \mathbb{Z}^d$.

[8] The proof of this statement will be published elsewhere. See also Ref. 26, where a similar fact was proved for the Schrödinger equation with a random potential, and Ref. 27, where it was done for elliptic operators of arbitrary order with random coefficients.

The statement formulated obviously holds for Eq. (2.12), and therefore

$$\chi(u) = \sup_{\zeta \geqslant 0} \{ f_G(\zeta) - \zeta u \}$$

Proceeding from this relation and Eqs. (2.5), (2.9), (2.16), (2.17), and (2.18) we come to the following principal statement of this section.

Let random exchange integrals $\mathcal{J}_{rr'}$ and the external field satisfy conditions (1)–(4). Then, if $N \to \infty$, with probability 1, there exists a thermodynamical limit of the free energy of the disordered spherical model, which is nonrandom and equals

$$f(\beta) = \begin{cases} \sup_{\zeta > 0} \{ f_G(\zeta) - \zeta \} - \dfrac{\mathcal{J}}{2}, & \beta \leqslant \beta_c \\ f_G(0) - \dfrac{\mathcal{J}}{2}, & \beta \geqslant \beta_c \end{cases} \tag{2.19}$$

where $f_G(\zeta)$ is given by Eq. (2.18), and β_c (inverse critical temperature) is the solution of equation

$$f'_G(0) = 1 \tag{2.20}$$

First, we are to show that in the ordered case, where $\mathcal{J}_{rr'}$ and h_r are nonrandom, the above obtained expressions transform into the known ones. It is convenient to use the following formulas[25]:

$$\nu(a) = \langle E_{00}(a) \rangle, \qquad \nu_1(a) = \left\langle \sum_{r \in \mathbb{Z}^d} E_{0r}(a) h_0 h_r \right\rangle \tag{2.21}$$

where $E_{rr'}(a)$ is the resolution of identity of self-adjoint operator $\hat{A}$ in $l_2(\mathbb{Z}^d)$ specified by matrix $\mathcal{J}\delta_{rr'} - \mathcal{J}_{rr'}(1 - \delta_{rr'})$. Since in the ordered case $\mathcal{J}_{rr'}$ are a function of difference $r - r'$, and h_r is independent of r, then after transition to the Fourier transform, $\hat{\mathcal{J}}$ and $\hat{A}$ are operators of multiplication by

$$\hat{\mathcal{J}}(q) = \sum_{r \in \mathbb{Z}^d} e^{iqr} \mathcal{J}_r, \qquad \tilde{A}(q) = \mathcal{J} - \tilde{\mathcal{J}}(q)$$

and therefore

$$\frac{d\nu}{da} = |B|^{-1} \int_B \delta(a - \tilde{A}(q))\, dq, \qquad \frac{d\nu_1}{da} = \delta(a - \tilde{A}(0)) h^2 \tag{2.22}$$

where $\mathcal{J} = \sup_{q \in B} \tilde{\mathcal{J}}(q)$, B is the first Brillouin zone of lattice $\mathbb{Z}^d$. If Eq. (2.22) is substituted into (2.19), we find that in conformity to Refs. 1 and 2, in the ordered case the spherical model free energy is

$$f(\beta) = -\frac{1}{2\beta} \ln \frac{2\pi}{\beta} + \frac{1}{2\beta|B|} \int_B \ln[2z - \hat{\mathcal{J}}(q)]\, dq - \frac{h^2}{2[2z - \tilde{\mathcal{J}}(0)]} - z \tag{2.23}$$

where z is the solution to equation

$$\frac{1}{\beta|B|}\int_B \frac{dq}{2z-\tilde{\mathcal{J}}(q)}+\frac{h^2}{[2z-\tilde{\mathcal{J}}(0)]^2}=1 \tag{2.24}$$

(see footnote 9). The above equation, in the ordinary derivation of Eq. (2.23), is obtained as the expression for determination of the steepest descent point. Write also the expression for the reciprocal critical temperature β_c following from Eq. (2.23), for $h=0$:

$$\beta_c=|B|^{-1}\int_B \frac{dq}{\mathcal{J}-\tilde{\mathcal{J}}(q)}=\int_0^\infty \frac{\nu'(a)\,da}{a} \tag{2.25}$$

The nonanalytical in β and h behavior of various thermodynamical values corresponding to phase transitions appearing in the model under discussion is the case for $\beta\uparrow\beta_c$, where the solution of Eq. (2.25) approaches $\mathcal{J}$ and is determined by the behavior of $\tilde{\mathcal{J}}(q)$ in the vicinity of this value.

In conclusion of this section, note that the main formulas (2.18) to (2.20) can be obtained also in the framework of the so-called mean spherical model as well.[28,29] This model, unlike the original one of Ref. 1, treats the spherical condition $\sum_{r\in V}s_r^2=N$ as fulfilled only in the average. That means that these two models are related to one another in the same way as the canonical and grand canonical ensembles of the statistical mechanics, the density in the former being equal for all microstates, while in the latter only the average for the ensemble is assigned. As was shown in Ref. 29, the models are equivalent for all values of thermodynamical parameters at which there are no phase transitions, or when an external field removing the transition (i.e., breaking the symmetry) is added to energy (2.1). This seems quite natural in terms of the mentioned similarity to canonical and grand canonical ensembles, since as is known,[23] thermodynamical ensembles are equivalent when there are no phase transitions in the system.

Note also that the principal equations (2.18)–(2.20) are valid wherever exist limits in Eqs. (2.16) and (2.17). The latter can be the case even though not all of the conditions (1)–(4) are satisfied. Let, e.g., $\mathcal{J}_{rr'}$ be equal to $-2\mathcal{J}$ for a certain pair (r,r') be zero for the rest of pairs and $h=0$. Then $\nu'(a)=\delta(a-\mathcal{J})$ and the associated free energy can be readily calculated. This expression was first obtained in Ref. 55, where moreover it was demonstrated to equal the free energy in the spherical model with the Curie–Weiss interaction, $\mathcal{J}_{rr'}=\mathcal{J}N^{-1}$. In our formalism this fact immediately follows from the circumstance that the $\nu'(a)$ corresponding to such an interaction is also equal to $\delta(a-\mathcal{J})$.

[9] In writing Eqs. (2.23) and (2.24), to compare them to the known formulas, we passed from variable ζ to z and from $\hat{A}$ to $\hat{\mathcal{J}}$.

3. SPHERICAL MODEL WITH NONRANDOM EXCHANGE INTEGRALS

We start with a brief discussion of some known results associated with the ordered spherical model. Remember that this is the case of the finite-range interaction, i.e., $\mathcal{J}_{r-r'} = 0$ with $|r - r'| \geqslant R_0$.[10] Let $q_0 \in B$ be the point where $\tilde{\mathcal{J}}(q)$ reaches the maximum. Then, if the Gaussian of the function $\tilde{\mathcal{J}}(q)$ is not degenerate at this point (obviously the general case), expansion of $\tilde{\mathcal{J}}(q)$ in the vicinity of q_0 starts with terms proportional to q^2 and therefore,

$$\nu'(a) = C_d a^{d/2-1}\left[1 + o(1)\right], \qquad a\downarrow 0 \tag{3.1}$$

where C_d is a constant. Hence, in the ordered spherical model the phase transition in the zero field appears at a dimension no less than $d = 3$, since β_c of Eq. (2.25) becomes finite for $d \geqslant 3$. As is suggested by Eq. (2.23), magnetization $m = -\partial f/\partial h$ is given by relation

$$m(h) = \frac{h}{2z(h) - \tilde{\mathcal{J}}(0)} \tag{3.2}$$

where $z(h)$ is the solution to Eq. (2.24). Thus, the mathematical mechanism of the appearance of the nonzero spontaneous magnetization, m_0, in the spherical model is a sufficiently rapid approach of $z(h)$ to $\tilde{\mathcal{J}}(0)$ when $h \to 0$. Such is the case with ferromagnets for which $\mathcal{J}_r \geqslant 0$ and therefore, $\mathcal{J} \equiv \sup_{q\in B} \tilde{\mathcal{J}}(q) = \tilde{\mathcal{J}}(0)$. Here when $\beta \geqslant \beta_c$

$$m_0 = (1 - \beta_c/\beta)^{1/2} \operatorname{sign} h \tag{3.3}$$

(see footnote 11) for all $d \geqslant 3$. As regards the magnetic susceptibility in zero field, it is finite and varies as $(\beta_c/\beta - 1)^{2/d-2}$ when $\beta < \beta_c$, and when $\beta > \beta_c$

$$\chi = \begin{cases} \infty, & d \leqslant 4 \\ \text{const}(\beta - \beta_c)^{-1}, & d > 4 \end{cases} \tag{3.4}$$

Thus, if $\mathcal{J}_2 \geqslant 0$, when $\beta = \beta_c$, the model has the phase transition to the ferromagnetic state characterized by a nonzero spontaneous magnetic moment. If a uniform external field is switched on, no matter how small it may be, the phase transition disappears.

Now take the case of nonpositive $\mathcal{J}_r$, which may appear in case of antiferromagnetic interaction ($\mathcal{J}_r \leqslant 0$), interaction due to indirect exchange (oscillating in $|r|\mathcal{J}_r$), and dipole, spin-orbital, or superexchange interaction

[10] One can readily make sure that the facts we are stating are true even when $\sum_{r\in\mathbb{Z}^d} r^2|\mathcal{J}_r| < \infty$. The case of slower decreasing interaction was analyzed in Ref. 2.

[11] We put $x_+ = x$ if $x \geqslant 0$ and $x_+ = 0$ if $x \leqslant 0$.

($\mathcal{J}_r$ changing its sign as the direction of r is changed).[30] In all these cases, if negative $\mathcal{J}_r$ are many enough, $\tilde{\mathcal{J}}(0) < \mathcal{J}$.[12] Since, by Eq. (2.24), when $\beta > \beta_c$, $2z(h)\downarrow\mathcal{J}$ with $h \to 0$, then, as appears from Eq. (3.2), there is no spontaneous magnetization in such systems. The magnetic structures that arise in such cases either include several sublattices with zero total magnetic moment, or are noncollinear and noncommensurable magnetic helical structures also possessing zero macroscopic moment (see Refs. 30 and 31 and Section 5 of this paper).

The magnetic susceptibility is in this case

$$\chi = \begin{cases} [\mathcal{J} - \tilde{\mathcal{J}}(0)]^{-1}, & \beta \geqslant \beta_c \\ [2z(0) - \tilde{\mathcal{J}}(0)]^{-1}, & \beta \leqslant \beta_c \end{cases} \tag{3.5}$$

where $z(0)$ is the solution to Eq. (2.24) for $h = 0$. As for $\beta \uparrow \beta_c$, $2z(0) - \mathcal{J}$ behaves as $(\beta_c - \beta)^\alpha$, $\alpha = \min(1, 2/d - 2)$ then for all d, χ in the vicinity of β_c is continuous, but when $d \geqslant 4$, its plot has a cusp at $\beta = \beta_c$, since

$$\frac{d\chi}{d\beta} = \begin{cases} 0, & \beta \geqslant \beta_c \\ -\text{const}, & \beta \uparrow \beta_c \end{cases} \tag{3.6}$$

We see that the behaviors of m and χ are only determined by the form of $\tilde{\mathcal{J}}(q)$ near its maximum and therefore these quantities are the same for so widely differing physical situations we described above, where because of $\mathcal{J}_r$ sign alternation, $\mathcal{J}$ and $\tilde{\mathcal{J}}(0)$ may differ. This is only when we examine the possible spin configurations, i.e., quantities m_r which are Gibbsian averages of s_r, that we see a difference (these questions will be discussed in Section 5). This accounts for the finiteness of the magnetic susceptibility representing a response to a uniform applied field, and for the resulting stability of the involved ordered state when a uniform external field is switched on: it does not vanish unless $h \geqslant \tilde{\mathcal{J}}(0)$ lower fields resulting only in decreasing critical temperature:

$$\beta_c^* = \beta_c\left(1 - h^2/\tilde{\mathcal{J}}(0)\right)^{-1} > \beta_c \tag{3.7}$$

The singularity at $\beta = \beta_c$ in both the cases appears in the response to the field proportional to $e^{iq_0 r}$ where q_0 is such that $\sup_{q \in B} \tilde{\mathcal{J}}(q) = \tilde{\mathcal{J}}(q_0)$ and therefore the ordered low-temperature state is absolutely unstable with respect to the field switch-on (it vanishes at arbitrarily small amplitude of the field).

Note that $e^{iq_0 r}$ is proportional to the eigenfunction of operator $\hat{\mathcal{J}}$ corresponding to the maximum spectral parameter value. Therefore, sum-

[12] For example, in a simple cubic lattice, for $\mathcal{J}_r = \mathcal{J}_1 > 0$, for nearest neighbors, $\mathcal{J}_r = \mathcal{J}_2 < 0$ if r are next-nearest neighbors, $\mathcal{J} > \tilde{\mathcal{J}}(0)$ if $\mathcal{J}_1/4|\mathcal{J}_2| < 1$.

ming up the above results, one sees that the thermodynamical properties of the spherical model entirely depend on the spectral properties of the operators $\hat{\mathcal{J}}_N$ and $\hat{\mathcal{J}}$ in the vicinity of the right-hand end of their spectrum.

Consider now the spherical model with nonrandom exchange integrals depending only on the $r - r'$ difference, but with random magnetic field h_r. Such models with nonrandom interaction and static quenched field are in a sense the simplest nontrivial examples of disordered spin systems and have recently been considered more than once.[(9–11,15,16),13]

In accordance with the general equations (2.19) and (2.20) of the preceding section, $f_G(z)$ should be found from Eq. (2.18). In this case, $f_1(z)$ obviously will be just equal to that in the ordered case, and

$$f_2(z) = -\frac{h^2}{2[2z - \tilde{\mathcal{J}}(0)]} - \frac{1}{2|B|}\int_B \frac{\tilde{W}(q)\,dq}{2z - \tilde{\mathcal{J}}(q)} \tag{3.8}$$

where $h = \langle h_r \rangle$ and $\tilde{W}(q)$ is the Fourier transform of the correlation function of h_r, i.e., function $W_r = \langle h_0 h_r \rangle - h^2$.

This formula shows that when $h \neq 0$, the thermodynamical properties of the model under discussion are similar to those of the homogeneous (ordered) spherical model in which $h_r \equiv h$. Therefore let us consider the case of $h = \langle h_r \rangle = 0$. The properties of such a model are essentially dependent on the space dimension d. Thus, for $d \leqslant 4$, the ferromagnetic state, which in case of $\mathcal{J}_r \geqslant 0$ and $d \geqslant 3$ exists when $\beta > \beta_c$ and $h_r = 0$, vanishes when an arbitrarily small field is switched on. Indeed, as follows from Eq. (3.8), for such instability to be the case, the derivative of function should go to infinity when $2z = \tilde{\mathcal{J}}(0) = \mathcal{J}$.

When $h \neq 0$, we find this property with the first term of Eq. (3.8). When $h = 0$ and the space dimension is only moderately high, the second term can also become singular. Indeed, let, e.g., the values of random field h_r be not correlated at various points, i.e., $W_r = \delta_{r0}$. Then $\tilde{W}(q) = \text{const}$ and

$$\frac{\partial f_2}{\partial z} = \begin{cases} [2z - \tilde{\mathcal{J}}(0)]^{2-d/2}, & d < y \\ -\ln[2z - \tilde{\mathcal{J}}(0)], & d = 4 \end{cases}$$

when $2z \downarrow \tilde{\mathcal{J}}(0)$. Clearly the same asymptotic behavior of f_2' will be the case wherever $\tilde{W}(0) \neq 0$. So, in all such cases the ferromagnetic state in the spherical model is unstable if the space dimension is not in excess of 4. This

[13] Some of the physical situations in which such fields appear are described in Refs. 9 and 15. A random field in the spherical model may also be generated by certain types of random exchange integrals. Take, for example, $\mathcal{J}_{rr'}$ in the factorized form $\mathcal{J}_{rr'} = \alpha_r \alpha_{r'}$, $\alpha_r = \pm 1$; in this case substitution $s_r \to \alpha_r s_r$ leads to the field $h_r = h\alpha_r$.[(12)]

was first shown in Ref. 9 for models with continuous symmetry by a heuristic analysis of the domain boundary energies and the linear susceptibility divergence. It was proved also in Ref. 16 for the X-Y model with Gaussian noncorrelated h_r; however, in the latter paper the so-called replica trick was used which is difficult to control.

One can also make sure that, when $d > 4$, the transition to the ferromagnetic state takes place if parameter

$$\mathcal{C} \equiv \frac{1}{|B|} \int_B \frac{\tilde{W}(q)\, dq}{\left[\tilde{\mathcal{J}}(0) - \tilde{\mathcal{J}}(q)\right]^2}$$

which is a certain measure of the random field fluctuations is less than 1, viz. $\mathcal{C} < 1$. Then the role of the transition temperature is played by quantity [cf. Eq. (3.7)]

$$\beta_c^* = (1 - \mathcal{C})^{-1} \beta_c > \beta_c$$

so that the switch-on of a random field with zero average value at $d > 4$ impedes the phase transition, which occurs at a lower temperature β_c^*, and even disappears when the field fluctuations are large enough and therefore $\mathcal{C} \geqslant 1$.

4. SPHERICAL MODEL WITH RANDOM EXCHANGE INTEGRALS

Now analyze the general formulas of Section 2 for the case of major importance to us, i.e., where exchange integrals $\mathcal{J}_{rr'}$ are random. As our primary interest is the phase transitions and changes therein induced by the homogeneous external field switch-on, we assume $h_r \equiv h$ for Eq. (2.1). In this case the function $f_2(\zeta)$ of Eq. (2.18) becomes

$$f_2(\zeta) = -\frac{h^2}{2} \varphi(\zeta)$$

$$\varphi(\zeta) = \lim_{N\to\infty} \left\langle N^{-1} \sum_{r,r' \in V} G_{rr'} \right\rangle = \int_0^\infty \frac{d\mu(a)}{2\zeta + a} \tag{4.1}$$

$$\mu(a) = \lim_{N\to\infty} N^{-1} \sum_{a_k \leqslant a} \left| \sum_{r \in V} \psi_k(r) \right|^2$$

These formulae and Eq. (3.19) show that the magnetic moment in the model under discussion is

$$m(h) = -\frac{\partial f}{\partial h} = h\varphi(\zeta(h))$$

where $\zeta(h)$ is the root of Eq. (3.20), which in this case becomes

$$\frac{1}{\beta}\int_0^\infty \frac{d\nu(a)}{2\zeta + a} + h^2\int_0^\infty \frac{d\mu(a)}{(2\zeta + a)^2} = 1 \tag{4.2}$$

The above formulas imply the following statements:

(a) the phase transition in a zero field takes place then and only then, when the integral

$$\int_0^\infty \frac{d\nu(a)}{a}$$

is finite.

(b) The spontaneous magnetic moment m_0 is zero below the critical temperature ($\beta > \beta_c$) then and only then, when $\mu(+0) = 0$ (see footnote 14).

(c) The phase transition does not disappear in response to the switch-on of an infinitely small uniform field then and only then, when the integral

$$\int_0^\infty \frac{d\mu(a)}{a^2} = \varphi'(0)$$

converges, for which it is sufficient that, when $a\downarrow 0$, $\mu(a)$ should tend to zero no slower than $a^{2+\epsilon}$, $\epsilon > 0$.

It appears from these statements that to gain an insight into the thermodynamical properties of the disordered spherical model, it is necessary to know the behaviors of the distribution function $\nu(a)$ and $\mu(a)$ in the vicinity of the origin, i.e., the boundary of the operator $\hat{\mathcal{J}}$ spectrum. However, unlike the ordered case, where, as we saw in the preceding section, these functions are calculable quite easily, it is far not so in the disordered case. These problems are given much consideration in the theory of disordered systems. Thus, for instance, even the studies of eigenvalue distribution functions comprise a considerable part of it. However, most works dealt with the Schrödinger equation with random potential and its discrete analogs, i.e., the case of operators with the so-called diagonal disorder (see, e.g., review, Refs. 19 and 23, and also Refs. 20, 33, and 34). The eigenvalue distribution of random operators with off-diagonal disorder, which appear in magnetic problems, is much less studied. Very little if at all

[14] Since by the very definition, $\mu(-0) = 0$, then naturally equality $m_0 = 0$ at $\beta > \beta_c$ is equivalent to the continuity of function $\mu(a)$ at $a = 0$. For example, in the ordered model discussed in the preceding section, in accordance with Eq. (2.22), $\mu'(a) = \delta(a - \tilde{A}(0))$ and therefore in case of alternating exchange integrals, where $\tilde{A}(0) > 0 = \inf_{q\in B}\tilde{A}(q)$ there was no spontaneous magnetic moment, as we saw.

is known also of function $\mu(a)$, particularly as concerns the non-one-dimensional situation.

Therefore we shall first dwell upon a comparatively simple case permitting us to get the required information by means of rather simple arguments, provided that special assumptions are made. Certain more general results for the asymptotic behavior of functions $\nu(a)$ and $\mu(a)$ when $a\downarrow 0$ are supplied in Appendixes A and C.

So, let us consider the case of ferromagnetic interaction of nearest neighbors, when

$$\mathcal{J}_{rr'} = \begin{cases} 0, & |r - r'| \neq \delta_l \\ \mathcal{J}_r^{(l)}, & r' = r + \delta_l \\ \mathcal{J}_{r-\delta_l}^{(l)}, & r' = r - \delta_l \end{cases} \tag{4.3}$$

where $\pm\delta_l$, $l = 1, 2, \ldots, d$ are vectors connecting the particular point of the simple cubic lattice with nearest neighboring second points.

Assume for the random $\mathcal{J}_r^{(l)}$ variables that all of them are independent, identically distributed and such that

$$\sup \mathcal{J}_r^{(l)} = K < \infty, \qquad \inf \mathcal{J}_r^{(l)} = k > 0 \tag{4.4}$$

and (which is the main technical assumption)

$$\Pr\{\mathcal{J}_r^{(l)} > K - \epsilon\} \leqslant C\epsilon^{\alpha}, \qquad \alpha > 1/d \tag{4.5}$$

Otherwise speaking, we require that random variables $\mathcal{J}_r^{(l)}$ should take on their maximum values with sufficiently small probability.

As is shown in Appendix B, under such conditions,

$$\int_0^{\infty} \frac{d\nu(a)}{a} < \infty, \qquad \int_0^{\infty} \frac{d\mu(a)}{a^2} < \infty \tag{4.6}$$

Hence, based on the above-formulated statements (a)–(c), we come to the following conclusions concerning the disordered spherical model with the interaction described: (1) The phase transition in the zero field is possible for any dimension of the space; (2) the low-temperature phase has no spontaneous magnetization; and (3) the transition does not disappear if a uniform external field is switched on.

One may think that the sufficiently rapid approach to zero of functions $\nu(a)$ and $\mu(a)$ when $a\downarrow 0$ and the ensuing conclusions (1)–(3) are due to the special character of conditions (4.5). However, as is shown in Appendix A, in the one-dimensional case, function $\nu(a)$ corresponding to random variables $\mathcal{J}_r$, assuming only two magnitudes $K > k > 0$, where the conditions

(4.5) are obviously not fulfilled, has the following asymptotics:

$$\ln \nu(a) = -\left(C/\sqrt{a}\right)\left[1 + o(1)\right], \qquad a \downarrow 0$$
$$C = |\ln p|\pi^2 K, \qquad p = \Pr\{\mathcal{J}_r = K\} \tag{4.7}$$

Thus, even in the one-dimensional case, $\nu(a)$ can tend to zero more rapidly than any power of a.

The results of many papers which studied the function $\nu(a)$ show that for operators with diagonal disorder, such behavior of this function is characteristic and can be observed in any space dimension and for a wide class of probability distributions of matrix elements, provided that the statistical correlations between elements are sufficiently weak. Such behavior is the case because the spectrum adjacent to the boundary of the type under consideration (usually termed fluctuational[10,19]) is due to rather rare fluctuations (large deviations) of matrix elements when these are close to a certain extremal value over a sufficiently large region of the lattice. The method used in Appendix A to obtain the asymptotic form of Eq. (4.7) evidences that in case of off-diagonal disorder such asymptotic behavior of $\nu(a)$ must also be largely universal.

Besides, we show in Appendix C that even conditions (4.5) are sufficient for the function $\mu(a)$ in any dimension to be continuous at zero.

What has been said suggests that the above formulated properties (1)–(3) of the disordered spherical model are rather general in nature. They however disagree with the present-day idea of disordered magnetic systems. Thus, for example, any reasonable model with positive interaction should display ferromagnetic phase transition. As regards the Ising model and the classical X-Y model, this follows, for example, from the Griffith inequalities.[23] Therefore it seems that the spherical model generalization to the disordered case in the above form is not a reasonable model of disordered spin systems.

Without attempting to build up a more plausible modification of the disordered spherical model in the general case of alternating exchange integrals (one of such modifications will be the subject of a special paper), we shall show here how such modification may be made in the simplest case of nonnegative $\mathcal{J}_{rr'}$ (ferromagnets). It is achieved by substituting for $-\mathcal{J}_{rr'}$ in the energy expression (2.1) the following quantities:

$$I_{rr'} = \delta_{rr'} \sum_{\rho \neq r} \mathcal{J}_{\rho r'} - (1 - \delta_{rr'})\mathcal{J}_{rr'} \tag{4.8}$$

and accordingly the energy expression (2.1) becomes

$$H' = \frac{1}{2} \sum_{r,r' \in V} I_{rr'} s_r s_{r'} - \sum_{r \in V} h_r s_r \equiv \frac{1}{2}(\hat{I}_N s, s) - (h, s)$$

In the Heisenberg model, where $s_r^2 = \text{const}$, or in the ordered spherical model, where $\mathcal{G}_{rr'}$ depends only on the difference $r - r'$, H and H' differ only by the additive constant and are therefore thermodynamically equivalent. But this is not so in the disordered case. To make certain of that, note first of all that since $I_{rr'}$ satisfy the conditions (1)–(4) of Section 1, if so do $\mathcal{G}_{rr'}$, then the free energy of our modified model can still be calculated by the general formulas (3.18)–(3.20), with $z + I/2$ instead of the variable ζ, where

$$I = \inf_{\|x\|=1} \sum_{r,r' \in \mathbb{Z}^d} I_{rr'} x_r \bar{x}_{r'} = \inf_{\|x\|=1} \sum_{r,r' \in \mathbb{Z}^d} \mathcal{G}_{rr'} |x_r - x_{r'}|^2, \qquad \|x\|^2 = \sum_{r \in \mathbb{Z}^d} |x_r|^2$$

An important property of $I_{rr'}$, which along with the nonnegativity of $\mathcal{G}_{rr'}$ is responsible for the more reasonable properties of the modified model than those of the former one, is expressed as

$$\sum_{r' \in \mathbb{Z}^d} I_{rr'} = 0 \tag{4.9}$$

It is the analog of the conservation of the total spin and, roughly, means that every random operator $\hat{I}$ determined in $l_r(\mathbb{Z}^d)$ by matrix $I_{rr'}$ has the constant as the eigenfunction corresponding to zero eigenvalue.

Even this equation (4.9) alone allows us to explicitly calculate function $\varphi(\zeta)$ in Eq. (4.1). Denote by d_N the vector of $l_2(V)$ with components $N^{-1/2}$, $r \in V$. Then $\varphi(\zeta)$ may be written as follows:

$$\varphi(\zeta) = \lim_{N \to \infty} (\hat{G}_N d_N, d_N)$$

or, because of the known identity for the Green function $\hat{G}_N = (2z + \hat{I}_N)^{-1}$ of operator $\hat{I}_N$, $\hat{G}_N = (2z)^{-1} - (2z)^{-2}\hat{I}_N + (2z)^{-2}\hat{I}_N^2\hat{G}_N$ as

$$\varphi(\zeta) = (2z)^{-1} - \lim_{N \to \infty} (2z)^{-2}(\epsilon_N, d_N) + \lim_{N \to \infty} (2z)^{-2}(\hat{G}_N \epsilon_N, \epsilon_N)$$

where $\epsilon_N = \hat{I}_N d_N$. But it follows from properties (1) and (4) of random variables $\mathcal{G}_{rr'}$ and Eq. (4.9), that the number of nonzero components of vector ϵ_N have order $N^{d-1/d} R_0$ and each of them has order $N^{-1/2}$. Therefore the norm of ϵ_N in $l_2(V)$ has order $N^{-1/d}$ and consequently, when $z > I/2$, both the limits on the right-hand side of the above relation are zero. Thus, finally,

$$\varphi(\zeta) = -\frac{1}{2(\zeta - I/2)} \tag{4.10}$$

i.e., $\varphi(\zeta)$ has the same form here as that in the ordered spherical model [cf. Eq. (2.23)].

In obtaining Eq. (4.10) we used only condition (4.9). Now take into account the ferromagnetic character of the interaction, i.e., the nonnegative character of the random variables $\mathcal{I}_{rr'}$. Owing to this property and to Eq. (4.9), point 0 is the lower boundary of the spectrum of the random operator $\hat{I}$, so that $I = 0$ and $z = \zeta$. However, the character of this boundary is now quite different. In virtue of Eq. (4.9), the spectrum in the vicinity of the boundary (which can be called stable) is due to typical realizations of and therefore the distribution function of the operator $\hat{I}$ eigenvalues is no more exponentially small at low a, in contrast to the analogous function of operator $\hat{\mathcal{I}}$. Moreover, as in the ordered case [cf. Eq. (3.1)],

$$\nu(a) \geqslant \bar{\nu}_d a^{d/2}\left[1 + o(1)\right], \qquad a\downarrow 0 \tag{4.11}$$

Indeed, it follows from the results of Ref. 25 that $\nu(a)$ will not change if elements $I_{rr'}$ in a fixed thickness layer adjacent to the boundary of V are changed in an arbitrary fashion. Therefore replace $I_{rr'}$ by

$$\tilde{I}_{rr'} = \delta_{rr'} \sum_{\rho \neq r'} \mathcal{I}_{r\rho} - (1 - \delta_{rr'})\mathcal{I}_{rr'}, \qquad r, r' \in V \tag{4.12}$$

Then the corresponding quadratic form will become

$$\sum_{r,r' \in V} \mathcal{I}_{rr'}|x_r - x_{r'}|^2$$

which implies that the eigenvalues of operator $\tilde{I}_N$ are monotonic functions of elements $\mathcal{I}_{rr'}$. Therefore if we replace every $\mathcal{I}_{rr'}$ by its maximum value, which is by virtue of properties (2) and (3) of Section 2 a nonrandom $(r - r')$-independent quantity

$$\sup_{\omega \in \Omega} \mathcal{I}_{rr'}(\omega) = \bar{\mathcal{I}}_{r-r'}$$

then the function $\bar{\nu}(a)$ of this difference operator will be related to $\nu(a)$ as

$$\nu(a) \geqslant \bar{\nu}(a) \tag{4.13}$$

and for small a have asymptotic behavior as that of Eq. (3.1).

If the random variables of $\mathcal{I}_{rr'}$ lying at least in one of the "diagonals," i.e., $\mathcal{I}_{r,r+r_0}$ for a certain r_0 and all $r \in \mathbb{Z}^d$, are bounded from below by a positive number, then applying a similar reasoning, one can also obtain the upper estimate for $\nu(a)$ of the same type:

$$\nu(a) \leqslant \boldsymbol{\nu}(a) \tag{4.14}$$

where $\boldsymbol{\nu}(a)$ is the distribution function of eigenvalues of the Toeplitz operator with elements

$$\mathbf{I}_{r-r'} = \inf_{\omega \in \Omega} \mathcal{I}_{rr'}(\omega)$$

and consequently, on the assumptions made,

$$\nu(a) \leqslant \nu_d a^{d/2}\left[1 + o(1)\right], \qquad a\downarrow 0 \tag{4.15}$$

(see footnote 15). From Eqs. (4.13)–(4.15) it follows that the thermodynamical properties of the modified spherical model (4.8) in the ferromagnetic case are quite similar to those in the corresponding ordered model, that is, when $d \geqslant 3$ there is the phase transition in it which is attended by the appearance of spontaneous magnetization and this transition disappears as an arbitrarily small uniform field is switched on. Besides, the transition is characterized by critical exponents just the same as those for the ordered case.[3]

As is shown by some numerical calculations, the situation is the same also in some cases when $\mathcal{J}_{rr'}$ may go to zero with nonzero probability. Thus, if $\mathcal{J}_{rr'} = \mathcal{J}_{r-r'} c_r c_{r'}$, where the nonrandom function is nonzero only for the nearest neighbors and random variables c_r, the "occupation numbers," assume values 1 and 0 with probabilities p and $1-p$, respectively, then asymptotic form of $\nu(a)$ is also

$$\nu(a) = (a/\mathcal{D})^{d/2}[1 + o(1)], \qquad a\downarrow 0$$

The coefficient $\mathcal{D}$ depending on concentration p is positive for all $p > p_c$, where, for example, for the bcc lattice, $p = 0.178$.[37] One may think that, when $p < p_c$, $\nu(a) \sim a^{\gamma}$ with $\gamma < d/2$. Thus, if we make a natural assumption[38] that low-lying eigenfunctions of operator $\hat{I}$ have the form of quasi-plane-waves which can be characterized with the "dispersion law" $\omega(k)$ and for small k

$$\omega(k) = \mathcal{D} k^2 + F k^4$$

then, if $F \neq 0$ for $p \lesssim p_c$,

$$\nu(a) \sim (a/F)^{d/4}$$

Hence it follows that in this case, if $d \leqslant 4$, there will be no phase transition and therefore at $p \downarrow p_c$ the critical temperature in the three-dimensional model should tend to zero. One can readily show that this tendency will be as $T_c(p) \sim \mathcal{D}^{1/2}$, so that, if in accordance with results of Ref. 39 assume that $\mathcal{D}(p) \sim (p - p_c)^{\alpha}$ $\alpha = 1.33$, then

$$T_c(p) \sim (p - p_c)^{\alpha/2}, \qquad p \downarrow p_c$$

Such behavior of the critical temperature means that in the model under consideration there should appear the concentration phase transition.

[15] In the one-dimensional case, with nearest-neighbor interaction, the asymptotic form of ■ may be found[35,36]:

$$\nu(a) = (2\pi)^{-1}(a < \mathcal{J}^{-1} >)^{1/2}[1 + o(1)], \qquad a\downarrow 0$$

5. ON THE MAGNETIZATION FIELD IN THE SPHERICAL MODEL

As was mentioned in Section 3, even in the ordered case, several different nonpositive functions $\mathcal{J}_{r-r'}$ lead to the same thermodynamics. However, the magnetic structure arising at $T < T_c$ is, generally speaking, different in such cases. This becomes clear after calculation of the Gibbs expectation[16] of the spin variable s_r. In the disordered system, whose realizations are not translationally invariant, $\langle s_r \rangle_G = m_r$ depends on r; this dependence being different for various realizations, m_r is a random field. In terms of this field one can give a classification of magnetic structures and express some thermodynamical quantities (see Refs. 40–42).

As is known, to have a nonzero $\langle s_r \rangle_G$ value at $T < T_c$, one should introduce a symmetry-breaking term into the Hamiltonian. The structure of this term may be revealed by thermodynamical considerations. Indeed, let h_r in Eq. (2.1) be equal to $\epsilon\psi_k(r)N^{1/2}$, where $\psi_k(r)$ is the eigenfunction of matrix $\hat{\mathcal{J}}_N$ corresponding to the kth eigenvalue λ_k arranged in nonincreasing order. Then, following in essential features the line of argument of Section 2, we find that the corresponding free energy is

$$f_\lambda(z) = \frac{1}{2\beta} \ln \frac{2\pi}{\beta} - f_1(z) - \frac{\epsilon^2}{2(2z - \lambda)} - z$$

where $\lambda = \lim_{N\to\infty} \lambda_k$ (see footnote 17), and z is the solution of equation $f_1' + \epsilon^2(2z - \lambda)^{-1} = 1$. Hence [cf. Eqs. (2.23) and (2.24)],

$$\frac{\partial f_\lambda}{\partial \epsilon} = -\epsilon(2z - \lambda)^{-1}, \quad \lim_{\epsilon \downarrow 0} \frac{\partial f_\lambda}{\partial \epsilon} = \begin{cases} (1 - \beta_c/\beta)_+^{1/2} \operatorname{sign} \epsilon, & \lambda = \mathcal{J} \\ 0, & \lambda < \mathcal{J} \end{cases} \tag{5.1}$$

Thus, we see that in the thermodynamical limit, the effect of perturbation $\epsilon N^{1/2} \sum_{r\in V} \psi_k(r) s_r$ does not vanish only provided that $\psi_k(r)$ is the eigenfunction of $\hat{\mathcal{J}}_N$ corresponding to the eigenvalue λ_k tending to $\mathcal{J}$, i.e., the right-hand edge of the operator $\hat{\mathcal{J}}$ spectrum. This implies that the term of such form should act as a symmetry-breaking perturbation.

Since, when $\epsilon > 0$, $f_{\mathcal{J}}$ is a smooth function of ϵ, then, according to the Griffith theorem,[(44)] with probability 1,

$$\frac{\partial f_{\mathcal{J}}}{\delta \epsilon} = \lim_{N\to\infty} N^{-1/2} \sum_{r\in V} \psi_k(r) \langle s_r \rangle_G, \qquad \lambda_k \uparrow \mathcal{J}$$

This relation, combined with Eq. (5.1), means that the role of the order parameter in the spherical model should be played by the projection of

[16] Gibbs averaging will be denoted by $\langle \cdots \rangle_G$.

[17] As follows from Refs. 25 and 43, any growth point of function $\nu(\mathcal{J} - a)$ is a limit point for $\hat{\mathcal{J}}_N$ matrices eigenvalues for $N \to \infty$.

function $m_r = \langle s_r \rangle_G$ on the eigenfunction $\psi_k(r)$ corresponding to the maximum eigenvalue, or to be more explicit, by

$$\lim_{\epsilon\downarrow 0} \lim_{N\to\infty} N^{-1/2} \sum_{r\in V} \psi_k(r) m_r \equiv m_{\mathcal{J}}, \qquad \lambda_k \uparrow \mathcal{J}$$

Based on this and on the completeness of the $\{\psi_k\}$ system, it is natural to expect that in the thermodynamical limit,

$$m_r = m_{\mathcal{J}} \lim_{N\to\infty} N^{1/2} \psi_k(r) \tag{5.2}$$

This relation can be proved for some cases. Indeed, when $\epsilon > 0$, the steepest-descent method shows that for any fixed $r \in \mathbb{Z}^d$ and $\lambda_k \uparrow \mathcal{J}$,

$$\lim_{N\to\infty} \left[\langle s_r \rangle_G - \frac{\epsilon}{2z - \lambda_k} N^{1/2} \psi_k(r) \right] = 0 \tag{5.3}$$

Where there exists $\lim_{N\to\infty} N^{1/2}\psi_k(r) \equiv \psi(r)$ Eq. (5.3) after having taken the limit $\epsilon\downarrow 0$ leads to Eq. (5.4). The said limit exists in the ordered case. Here $\psi(r) = \cos q_0 r$, where q_0 is specified by relation $\sup_{q\in B} \tilde{\mathcal{J}}(q) = \tilde{\mathcal{J}}(q_0)$ and we obtain the known result that in the ordered spherical model [cf. Eq. (3.3)],

$$m_r = \operatorname{sign} \in (1 - \beta_c/\beta)_+^{1/2} \cos q_0 r$$

Depending on the lattice and interaction types, the spin distribution may widely vary.[30,31]

The question of existence of the above limit in the disordered case has received almost no attention. If we assume nevertheless that it exists, even though in the following weak sense

$$\lim_{N\to\infty} N^{-1} \sum_{r\in V} N^{1/2} \psi_k(r) \equiv A, \qquad \lambda_k \uparrow \mathcal{J} \tag{5.4}$$

(see footnote 18) then the absence of spontaneous magnetization even in case of nonnegative $\mathcal{J}_{rr'}$ stated in Section 4 finds a heuristic explanation. Indeed, if the left-hand limit in Eq. (5.4) exists, then the spontaneous magnetization is

$$m_0 = \lim_{N\to\infty} N^{-1} \sum_{r\in V} m_r = m_{\mathcal{J}} A$$

and thus its zero value means that $A = 0$. But if $\mathcal{J}_{rr'} \geqslant 0$, then according to the Perron–Frobenius theorem,[45] the eigenfunction $\psi_0(r)$ corresponding to the maximum eigenvalue does not change its sign anywhere and the zero value of A is only possible owing to the fact that $\psi_0(z)$, when $N\to\infty$, remains, roughly, a decreasing function. Such behavior is quite natural in

[18] It follows from the Cauchy inequality that the expression under the limit sign is bounded.

terms of the modern disordered system theory[21] in which it is widely accepted that the operator $\hat{\mathcal{G}}$ in the vicinity of the fluctuation boundary has the pure point spectrum.

It has been long known[2,46] that the spherical model has much in common with the ideal Bose gas, in particular the phase transition in it is analogous to the Bose–Einstein condensation. It is therefore no wonder that the thermodynamical behavior of the ideal Bose gas in a random field is characterized by equally unusual properties: the phase transition appears to be possible even in the one-dimensional case[47] and it is not accompanied by macroscopic occupation of the ground state (this latter fact is the analog of the absence of spontaneous magnetization in the spherical model), so that this transition may hardly be called a condensation.

In Ref. 48 a simple model of the one-dimensional Schrödinger equation with a field formed by point chaotically distributed scatterers of large intensity is considered to show that $\psi_0(r)$ is concentrated in a range of length $\ln N$, whence follows the impossibility of macroscopic condensation in the ground state.

The essence of our arguments has, as is seen, the same nature as those in Refs. 47 and 48.

From this point of view it also becomes clear why the spontaneous magnetization in the modified spherical model (4.9) is nonzero. Indeed, the reasons used in the derivation of Eq. (4.10) essentially imply that vector d_N in case of large N is in a sense close to the eigenvector of matrix $\hat{I}_N$ corresponding to the minimum eigenvalue, and then $N^{1/2}\psi_0 \sim N^{1/2}d_N = 1$.

The facts mentioned are ultimately due to the absence of the translational invariance in every realization of the disordered model [condition (2) of Section 2 provides such invariance in the mean only]. This leads to the possibility (first noted in Ref. 8) of much larger, than in the above case, fluctuations of the magnitudes of spin variables s_r and, as a consequence, to much larger, than in the ordered case, differences of the spherical model from the Heisenberg model where it is fixed. The existence of such fluctuations is indirectly evidenced by impossibility to obtain the disordered spherical model from the classical Heisenberg model in the infinite spin dimension limit (see Ref. 49 and Appendix D).

More straightforward evidence is as follows. Calculate, by the steepest-descent method, the $\langle s_r^2 \rangle_G$ value for $T > T_c$ where it can be done even in the disordered case. The result is

$$\lim_{N \to \infty} \langle s_r^2 \rangle_G = \beta^{-1} G_{rr}$$

where $\hat{G} = (2\zeta = \hat{A})^{-1}$, and since at $T > T_c$ $\zeta > 0$ and $\hat{A} \geqslant 0$, then the Green function, $\hat{G}$, is well defined. In the ordered case, $\hat{G}$ is a difference operator, G_{rr} does not depend on r and coincides, as is seen from Eq. (2.21), with the left-hand side of Eq. (2.24) for $h = 0$. Therefore for all T, $\beta^{-1}G_{rr} = 1$, that is, at least $\langle s_r^2 \rangle_G$ is in the ordered spherical model the same as in the Heisenberg model. In the disordered case the behaviors of the sum $N^{-1}\sum_{r\in V}\langle s_r^2\rangle_G$ and of an individual term are essentially different. While the former value, for $N \to \infty$, becomes nonrandom and therefore equal to $\langle\langle s_r^2\rangle_G\rangle$, which is equal to 1 because of Eq. (2.20), $\langle s_r^2\rangle_G$ on the other hand at $T \downarrow T_c$ in various realizations may have widely varying values. Indeed, write G_{rr} as

$$G_{rr} = \int_0^\infty \frac{d\nu_r(a)}{2\zeta + a} \tag{5.5}$$

where $\nu_r(a) = E_{rr}(a)$ and $\hat{E}(a)$ is the resolution of identity of the operator $\hat{A}$. Suppose that in the neighborhood of zero the spectrum of $\hat{A}$ has a point component (this fact is equivalent to the above-mentioned localization of disordered system eigenfunctions and may be rigorously proved for the one-dimensional case[50,51]). If $\mathcal{J}_{rr'} \geqslant 0$, then with using the Perron–Frobenius theorem, one can show that point 0 with probability 1 cannot be an eigenvalue and is only the limit point for them (this follows from the results of Ref. 43). Since, based on Eqs. (5.5), (2.21), and (4.7), $\lim_{\zeta\downarrow 0}\langle \beta^{-1}G_{rr}\rangle$ exists and is finite, then because of the monotonic dependence of G_{rr} on ζ [see Eq. (5.5)] $\lim_{\zeta\downarrow 0}\beta^{-1}G_{rr}$ exists and is finite with probability 1. However, the magnitude of this limit can become arbitrarily high, so far as the eigenvalues of $\hat{A}$ may approach point 0 arbitrarily closely. But as follows from Refs. 18 and 19, these eigenvalues are due to such essentially inhomogeneous fluctuations of $\mathcal{J}_{rr'}$, when they assume near-maximum values over a sufficiently large lattice domain.

Now consider in brief the question of the form of m_r in the spherical model with random field h_r which was discussed in Section 3. For simplicity, assume that h_r are statistically independent random variables. Because, when $h \equiv \langle h_r \rangle \neq 0$, Eq. (2.20) with $f_2(z)$ of Eq. (3.8) has the solution $z > \mathcal{J}/2$, then, by the steepest-descent method, we find that

$$m_r = \sum_{r'\in\mathbb{Z}^d} G_{rr'}h_{r'}, \qquad G_r = \frac{1}{|B|}\int_B \frac{e^{iqr}\,dq}{2z - \tilde{\mathcal{J}}(q)} \tag{5.6}$$

and the series involved here converges with probability 1 because of decrease in G_r, exponential when $2z > \tilde{\mathcal{J}}(0)$, and in accordance with the three series criterion.[52] The macroscopic magnetization is in this case

$$m = \langle m_r \rangle = h \sum_{r\in\mathbb{Z}^d} G_r = \frac{h}{2z - \tilde{\mathcal{J}}(0)} \tag{5.7}$$

in agreement with the result of differentiation of Eq. (3.8) with respect to h [cf. Eq. (3.2)]. When $d \leqslant 4$, $\lim_{h\to 0} z > \tilde{\mathcal{J}}(0)$ (see Section 3), and therefore the spontaneous magnetization is zero, in agreement with the ferromagnetic state instability established in Section 3. When $d > 4$, $m|_{h=0} \neq 0$, but the series in Eq. (5.6) converges much slower. Indeed, the mathematical expectation of the series terms is zero ($h = 0$) and the variance is proportional to G_r^2, and as, when $2z = \tilde{\mathcal{J}}(0)$, $G_r \sim |r|^{-(d-2)}$, then the convergence of the series of variances sufficient for the convergence of the series into Eq. (5.6) with probability 1 is provided only by inequality $d > 4$:

$$\sum_{r \in \mathbb{Z}^d} G_r \sim \int^{\infty} \frac{r^{d-1}\, dr}{r^{2(d-2)}} = \int^{\infty} \frac{dr}{r^{d-3}} < \infty, \qquad d > 4$$

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APPENDIX A. DISTRIBUTION OF EIGENVALUES OF RANDOM JACOBIAN MATRICES AND THE DISORDERED SPHERICAL MODEL OF ONE-DIMENSIONAL AND LAYERED MAGNETS

Consider the simplest case of disordered spherical model, assuming that the space is one dimensional, that the interaction involves only nearest neighbors, and the corresponding exchange integrals $\mathcal{J}_k$ are independent random variables taking on either of two values $0 < k < K$ with probabilities $1 - p$ and p, respectively.[19] In accordance with Eq. (2.18), for the construction of thermodynamics in a zero external field it is sufficient to know the distribution function of eigenfunctions of the Jacobian matrices $\hat{\mathcal{J}}$ with elements $\mathcal{J}_{mn} = \delta_{m+1,n}\mathcal{J}_n + \delta_{mn+1}\mathcal{J}_{n+1}$. Since the spectrum $\hat{\mathcal{J}}$ occupies the range $(-2K, 2K)$, then when changing from $\hat{\mathcal{J}}$ to $\hat{A} = \mathcal{J} - \hat{\mathcal{J}}$ $\mathcal{J} = 2K$, we come to the problem of finding the limit of Eq. (2.15) for a sequence A_N of matrices, corresponding to equations

$$-\mathcal{J}_{n+1} u_{n+1} + \mathcal{J} u_n - \mathcal{J}_n u_{n-1} = a u_n, \qquad n = 1, \ldots, N, \qquad u_0 = u_N = 0 \tag{A.1}$$

We shall refer to a segment of the chain as regular if at all its points, $\mathcal{J}_n = K$, and let $\mathfrak{N}_N(l)$ be the number of chain segments of length l which

[19] One can readily show that in the present case $\mathcal{J}_n$ may always be assumed to be nonnegative, so that the condition $\mathcal{J}_n \geqslant 0$ is not a limitation.

can be arranged, without overlapping, within the regular segments of the particular realization, which are contained in the range $(1, N)$. Then, following the reasoning of Ref. 36, we arrive at the following inequality:

$$N^{-1}\mathfrak{N}_N\left(\left[\frac{\pi}{\alpha}\right]\mathcal{G}+1\right) \leqslant \nu_N(a) \leqslant N^{-1}\mathfrak{N}_N(n_c+1)$$

where $a = \mathcal{G}(1-\cos\alpha)$, $n_c = [\pi/\alpha - 2\gamma]$, γ is a fixed number larger than $K^2 + k^2/K^2 - k^2$. By averaging this inequality over all realizations and noting that $a{\downarrow}0$ is equivalent to $\alpha{\downarrow}0$ we find that

$$\ln\nu(a) = -|\ln p|\left(\frac{\pi^2\mathcal{G}}{2}\right)^{1/2} a^{1/2}[1+o(1)], \qquad a{\downarrow}0 \tag{A.2}$$

which is the same as Eq. (4.7).

From this relation, the existence of a nonzero critical temperature immediately follows and therefore that of the phase transition in this simplest one-dimensional model. Show also that in this model the heat capacity and the derivative of the magnetic susceptibility with respect to temperature have a jump at $T = T_c$.

Indeed, from Eq. (2.19) it follows that

$$c = -T\frac{\partial^2 f}{\partial T^2} = \frac{1}{2} - \frac{\partial\zeta}{\partial T}$$

But at $T < T_c$ $\zeta = 0$; therefore $c = 1/2$. On the other hand, according to Eq. (4.2),

$$\left.\frac{\partial\zeta}{\partial T}\right|_{T=T_c+0} = T_c^{-2}\left(\frac{1}{2\mathcal{G}^2} + \int_0^{2\mathcal{G}} \frac{\nu\, da}{a^3}\right)^{-1}$$

and therefore $c(T_c+0) < 1/2$. Moreover, by using the same equation (4.2) and the Cauchy inequality, one can readily show that $\partial\zeta/\partial T < 1/2$, and therefore $c(T) \geqslant 0$ for all T.

Further, according to Eq. (2.18),

$$\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0} = f_1(\zeta|_{h=0})$$

provided that $\int_0^{2\mathcal{G}} \nu a^{-3}\,da < \infty$, $\int_0^{2\mathcal{G}} \mu a^{-4}\,da < \infty$ (see footnote 20). According to Eq. (2.18), f_1' is continuous at zero and negative. Therefore, by the same arguments as above, we obtain

$$\frac{\partial\chi}{\partial T} = \begin{cases} 0, & T = T_c - 0 \\ -\infty < \text{const} < 0, & T = T_c + 0 \end{cases}$$

[20] As was mentioned in Section 4, $\mu(a)$ also goes to zero exponentially when $a{\downarrow}0$; however the proof of this fact would require a different approach, and we shall omit it.

As in the case of ordered antiferromagnet (see Section 3), such behavior of χ is due to regularity of $f_1(\zeta)$ near zero, though in the ordered case $f_1(\zeta)$ is analytical, while in the disordered case it is only infinitely differentiable.

The technique to obtain the asymptotic form of $\nu(a)$ we have described also allows us to consider non-one-dimensional models which are disordered in one dimension only. The simplest two-dimensional model of this sort analogous to the Ising model considered in Ref. 53 is characterized by an interaction of the following form:

$$\mathcal{J}_{mn,m'n'} = \frac{B}{2}(\delta_{m+1,m'} + \delta_{m,m'+1}) + \mathcal{J}_n \delta_{mm'}(\delta_{n+1,n'} + \delta_{n,n'+1})$$

Since in the corresponding equation for eigenvalue and eigenvector determination, the variables are separated, then

$$\nu(a) = \frac{1}{\pi}\int_0^a \nu_0(a - \mathcal{C})\, d\mathcal{C}/(2B - \mathcal{C})^{1/2} \tag{A.3}$$

where $\nu_0(a)$ is the eigenvalue distribution function of the one-dimensional operator (A.1). From Eq. (A.3) it follows that when $a \downarrow 0$,

$$\nu(a) \leqslant \nu_0(a)(2a)^{1/2}/K$$

and therefore, in virtue of Eq. (A.2), $\nu(a)$ in the present two-dimensional case also exponentially goes to zero at the spectrum boundary. Therefore in particular, $\partial\chi/\partial T$ and the heat capacity at $T = T_c$ have a jump, whereas in the corresponding Ising model the heat capacity is a smooth function in the vicinity of $T = T_c$.

Similar bounds may be obtained also for the more complex d-dimensional case considered (by a different method) in Ref. 8, where

$$\mathcal{J}_{rr'} = (\mathcal{J}_m \delta_{m+1,m'} + \mathcal{J}_{m-1}\delta_{m-1,m'})\delta_{\rho\rho'} + \bar{\mathcal{J}}_m \delta_{mm'} \sum_{k=1}^{d-1} (\delta_{\rho+\delta_k,\rho'} + \delta_{\rho-\delta_k,\rho'})$$

where $r = (\rho, m)$, $\rho \in \mathbb{Z}^{d-1}$, $m \in \mathbb{Z}^1$,

$$\begin{aligned} \mathcal{J}_m &= \mathcal{J}_A(1 - \xi_m)(1 - \xi_{m+1}) + \mathcal{J}_B \xi_m \xi_{m+1} \\ &\quad + \mathcal{J}_{AB}\left[(1 - \xi_m)\xi_{m+1} + \xi_m(1 - \xi_{m+1})\right] \\ \bar{\mathcal{J}}_m &= \mathcal{J}_A(1 - \xi_m) + \mathcal{J}_B \xi_m \end{aligned}$$

and ξ_m are independent random quantities assuming values 0 and 1 with probability p and $1 - p$. This form of exchange integrals corresponds to layer magnets where the nearest-neighbor interaction in each $(d-1)$-dimensional layer is either $\mathcal{J}_A$ or $\mathcal{J}_B$ (A and B layers), but the arrangement of the layers is random (it is given by a sample of ξ_m values), and the spin interaction between neighboring layers of dissimilar type is $\mathcal{J}_{AB}$.

In this case

$$\nu(a) = \frac{1}{\pi^{d-1}} \int_0^{\pi} d\varphi^{d-1} \nu_\varphi(a) \tag{A.4}$$

where $\nu_\varphi(a)$ is the distribution function of eigenvalues of the one-dimensional Jacobian matrix with elements

$$\mathcal{I}_m \delta_{m+1,m'} + \mathcal{I}_{m-1} \delta_{m-1,m'} + \bar{\mathcal{I}}_m \delta_{mm'} \left(2 \sum_{k=1}^{d-1} \cos \varphi_k \right)$$

Therefore, based on Eqs. (1.2) and (1.4) we have

$$\nu(a) \leqslant C_1 a^{(d-1)/2} \exp(-C_2 a^{-1/2}), \qquad a \downarrow 0, \qquad C_1, C_2 > 0$$

APPENDIX B. PROOF OF EQ. (4.6)

The relations of Eq. (4.5) are apparently equivalent to the following ones:

$$\int_0^\infty N(t)\, dt < \infty, \qquad \int_0^\infty t M(t)\, dt < \infty$$

where

$$N(t) = \int_0^\infty e^{-at}\, d\nu(a), \qquad M(t) = \int_0^\infty e^{-at}\, d\mu(a) \tag{B.1}$$

From Eq. (2.21) and the definitions (2.15) and (4.1) of functions $\nu(a)$ and $\mu(a)$, it follows that

$$N(t) = \langle P_{00}(t) \rangle, \qquad M(t) = \left\langle \sum_{r \in \mathbb{Z}^d} P_{0r}(t) \right\rangle \tag{B.2}$$

where $P_{rr'}(t)$ is the kernel of operator $e^{-t\hat{A}}$ in $l^2(\mathbb{Z}^d)$. Write $A_{rr'}$ as

$$A_{rr'} = I_{rr'} + q_r \delta_{rr'}$$

where $I_{rr'}$ are defined in Eq. (4.8) and $q_r = \mathcal{I} - \sum_{r' \in \mathbb{Z}^d} \mathcal{I}_{rr'} \geqslant 0$. In virtue of nonnegativeness of $\mathcal{I}_{rr'}$ and the condition (4.9), operator $\hat{I}$ is an infinitesimal operator of the discontinuous Markov process $r(s)$, $s \geqslant 0$, $r(0) = r'$ in $\mathbb{Z}^d$ (Ref. 52) (the inhomogeneous Poisson process). Let $p_{rr'}(t)$ be its transition probability (it is the solution of equation $\partial p / \partial t = -Ip$, $p|_{t=0} = \delta_{rr'}$ and $E_r\{\cdots\}$ is the mathematical expectation operation it generates. Then by the Feynman–Kac formula[54] we have

$$P_{rr'}(t) = p_{rr'}(t) E_r \left\{ \exp\left[-\int_0^t q_{r(s)}\, ds \right] \middle| \, r(t) = r' \right\} \tag{B.3}$$

It immediately follows from this representation that $P_{rr'}(t) \geqslant 0$ and consequently according to Eq. (B.2), $0 \leqslant N(t) \leqslant M(t)$. Therefore it is sufficient to make sure that $M(t)$ decreases more slowly than $t^{2+\epsilon}$, $\epsilon > 0$. But according to Eqs. (B.2) and (B.3).

$$M(t) = \left\langle E_0 \left\{ \exp\left[-\int_0^t q_{r(s)}\, ds \right] \right\} \right\rangle$$

Because of the Jensen inequality,

$$\exp\left[-\int_0^t q_{r(s)}\, ds \right] = \exp\left(-t^{-1} \int_0^t tq\, ds \right) \leqslant t^{-1} \int_0^t ds \exp(-tq_{r(s)})$$

and then

$$M(t) \leqslant t^{-1} \int_0^t ds \langle E_0[\exp(-tq_{r(s)})] \rangle = t^{-1} \int_0^t ds \left\langle \sum_{r \in \mathbb{Z}^d} \exp(-tq_r) p_{0r}(s) \right\rangle \tag{B.4}$$

But it follows from the property of the space homogeneity in the mean [property (2), Section 2], the symmetry of function $p_{rr'}(t)$ and the normalization condition $\sum_{r' \in \mathbb{Z}^d} p_{rr'}(t) = 1$ that

$$\left\langle \sum_{r \in \mathbb{Z}^d} e^{-tq_r} p_{0r}(s) \right\rangle = \left\langle e^{-tq_0} \sum_{r \in \mathbb{Z}^d} p_{-r,0}(s) \right\rangle = \langle e^{-tq_0} \rangle$$

After substitution of this relation into Eq. (B.4) we arrive at the following inequality:

$$M(t) \leqslant \langle e^{-tq_0} \rangle = \langle e^{-t(K - \mathcal{J}_r)} \rangle^{2d} \tag{B.5}$$

which takes into account that in the case of the nearest-neighbor interaction considered in Section 4,

$$q_r = \sum_{\pm\delta} (K - \mathcal{J}_{r+\delta})$$

and $\mathcal{J} = \|\hat{\mathcal{J}}\| = 2dK$ in case of independent $\mathcal{J}_r$. Since according to condition (4.5),

$$\langle e^{-t(K - \mathcal{J}_r)} \rangle \leqslant O(t^{-\alpha}), \qquad t \to \infty, \qquad \alpha > d^{-1}$$

then it is just what follows from Eq. (B.5) that integrals in Eq. (4.6) converge.

APPENDIX C. PROOF OF THE CONTINUITY OF FUNCTION $\mu(a)$ AT ZERO

The sought continuity is obviously equivalent to relation $\lim_{t \to \infty} M(t) = 0$, where $M(t)$ is the Laplace transform of $\mu(a)$ entering into Eq. (B.1). The representation of $M(t)$ [Eq. (B.2)] and the boundedness of $P_{rr'}(t)$ [see

Eq. (B.3)], show that it is sufficient to prove equality

$$\lim_{t\to\infty} P_r(t) = 0$$

where $P_r(t) = \sum_{r'\in\mathbb{Z}^d} P_{rr'}(t)$, for almost every ω (i.e., for almost every sample of random $\mathcal{J}_{rr'}$ variables). But it follows from condition (2) of Section 2 and Eq. (B.3) that $P_r(t)$ has the following properties:

(i) $0 \leqslant P_r(t) \leqslant 1$.
(ii) $P_r(t, T_a\omega) = P_{r+a}(t,\omega)$, $a \in \mathbb{Z}^d$ i.e., $P_r(t,\omega)$ for every $t > 0$ is a homogeneous and ergodic random field [cf. Eq. (2.3)].
(iii) $\partial P/\partial t = -IP - qP$, $P|_{t=0} = 1$.
(iv) For every $r \in \mathbb{Z}^d$, $P_r(t)$ is a nondecreasing function of t (remember that $q_r \geqslant 0$).

Denote $\lim_{t\to\infty} P_r(t)$ existing by condition (iv) by π_r. Since from the boundedness of $\mathcal{J}_{rr'}$ and property (iii) follows the boundedness of the derivative $\partial P/\partial t$, and hence property (iv) implies its zero value in the limit $t = \infty$, then π_r satisfy equation

$$I\pi + q\pi = 0 \tag{C.1}$$

Apply operation $\lim_{N\to\infty} N^{-1}\sum_{r\in V}$ to this equation. Allowing for Eq. (4.9) and the homogeneity of random fields q_r and π_r, we find that $\langle q_r\pi_r\rangle = 0$. Since q_r and π_r are nonnegative, it follows that at almost every ω, for all $r \in \mathbb{Z}^d$,

$$q_r(\omega)\pi_r(\omega) = 0 \tag{C.2}$$

which together with Eq. (C.1) means that π_r satisfies equation

$$I\pi = 0 \tag{C.3}$$

i.e., is a "harmonic" function. Show that in this case the analog of the Liouville theorem is valid, i.e., that π_r is independent of r. To do so, multiply Eq. (C.3) by π_r and apply operation $N^{-1}\sum_{r\in\mathbb{Z}^d}$ to the result. This leads to relation

$$\left\langle \sum_{r'\in\mathbb{Z}^d} \mathcal{J}_{rr'}(\pi_r - \pi_{r'})^2 \right\rangle = 0, \qquad \forall r \in \mathbb{Z}^d$$

which because of nonnegativeness of $\mathcal{J}_{rr'}$ is equivalent to equality $\pi_r = \pi_{r'}$ for all those pairs (r, r') for which $\mathcal{J}_{rr'} > 0$. Therefore, if such pairs may connect any two lattice points, then $\pi_r = \text{const}$. Such a situation is obviously, for example, realized in the case of interaction of nearest neighbors and positive exchange integrals (ferromagnetic bond disorder) considered in Section 4.

But if $\pi_r(\omega)$ does not depend on r, then from Eq. (C.2) it follows that it is zero because, with probability 1, $q_r(\omega)$ is not zero for all r [otherwise, $q_r(\omega)$ would be a nonrandom constant].

APPENDIX D. FREE ENERGY OF THE ONE-DIMENSIONAL DISORDERED HEISENBERG MODEL IN THE INFINITE SPIN DIMENSIONALITY LIMIT

The statistical sum of such (classical) model is

$$Z_N = \int_{\|s_i\|=n} \exp\left(\beta \sum_1^N \mathcal{J}_k s_k s_{k+1} \right) \prod_1^N ds_k$$

where $\mathcal{J}_k$ are random variables satisfying conditions (1)–(3) of Section 2, s_k, n-dimensional vectors, and we assume that $s_1 = s_{N+1}$. By integrating successively over each s_k, we find that[3]

$$Z_N = \prod_{k=1}^N z_n(\beta n^{1/2} \mathcal{J}_k), \qquad z_n(x) = \int_{\|s\|=r} e^{xas}\, ds$$

where a is an arbitrary unit vector. Conditions (1)–(3) of Section 2 enable us to apply the ergodic theorem to $\ln Z_N$ and therefore when $N \to \infty$ the free energy of the model at hand, with probability 1, tends to value[21]

$$f_{\text{dis}}^H = -\beta^{-1} \left\langle \ln z_n(\beta \mathcal{J}_k n^{1/2}) \right\rangle$$

or

$$f_{\text{dis}}^H = \left\langle f_{\text{or}}^H(\mathcal{J}) \right\rangle_{\mathcal{J}}$$

where $f_{\text{or}}^H(\mathcal{J})$ is the free energy of the ordered chain of classical Heisenberg spins with the exchange integral equal to $\mathcal{J}$, i.e., $-\beta^{-1} \ln z_n(\beta \mathcal{J} n^{1/2})$.

Thus the transition from the ordered to the disordered model is carried out very simply in this case, by averaging the free energy of the ordered model over all possible values of the random exchange integral.

When calculating the integral entering into the definition of f_{or}^H for $n \to \infty$ by the Laplace method, we see that[1]

$$\lim_{n\to\infty} n^{-1} f_{\text{or}}^H = f_{\text{sp}}(\mathcal{J}) = -\frac{1}{2} \ln 2\pi - \frac{1}{2}\left[1 + (\beta\mathcal{J})^2\right]^{1/2} + \frac{1}{2} \ln \frac{1 + \left[1 + 4(\beta\mathcal{J})^2\right]^{1/2}}{2}$$

[21] This fact is a special case of the statement proved in Ref. 42 which says that this property of "selfaverageness" belongs to any disordered system with parameters satisfying conditions (1)–(4) of Section 2.

where f_{sp} is the free energy of the one-dimensional ordered spherical model with nearest-neighbor interaction. Therefore,

$$\lim_{n\to\infty} n^{-1} f_{dis}^{H} = \langle f_{sp}(J) \rangle_{\mathcal{J}} \tag{D.1}$$

But the right-hand side of this relation cannot be equal to the disordered spherical model free energy for the one-dimensional case as calculated in Section 4. Indeed, as was shown in Appendix A, even when independent $\mathcal{J}_k$ assume two values, $k/2$, $K/2$, the latter model has a phase transition due to exponentially decreasing to zero of the function $\nu(a)$ when $a\downarrow 0$. On the other hand, for such the right-hand side of Eq. (D.1) is as follows:

$$p f_{sp}\left(\frac{K}{2}\right) + (1-p) f_{sp}\left(\frac{k}{2}\right)$$

and since in the one-dimensional ordered spherical model there is no phase transition, this expression does not lead to a transition either.

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